

NORMALIZATION OF CLOSED EKEDAHL-OORT STRATA

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ABSTRACT. We apply our theory of partial flag spaces developed in [GK] to study a group-theoretical generalization of the canonical filtration of a truncated Barsotti-Tate group of level 1. As an application, we determine explicitly the normalization of the Zariski closures of Ekedahl-Oort strata of Shimura varieties of Hodge-type as certain closed coarse strata in the associated partial flag spaces.

INTRODUCTION

Let H be a truncated Barsotti-Tate group of level 1 (in short BT1), over an algebraically closed field k of characteristic p , and let $\sigma : k \rightarrow k$ denote the map $x \mapsto x^p$. Denote by $D := \mathbf{D}(H)$ its Dieudonné module, which is a finite-dimensional k -vector space D endowed with a σ -linear endomorphism F and a σ^{-1} -linear endomorphism V satisfying $FV = VF = 0$ and $\text{Im}(F) = \text{Ker}(V)$. Oort showed in [Oor01] that there exists a flag of D which is stable by V and F^{-1} and which is coarsest among all such flags, called the canonical filtration of D . After choosing a basis of D , we obtain a filtration of k^n (where $n = \dim_k(D)$ is the height of H). The stabilizer of this flag is a parabolic subgroup $P(H) \subset GL_n$, well-defined up to conjugation. We want to emphasize that this construction attaches a group-theoretical object $P(H)$ to a truncated Barsotti-Tate group of level 1.

The theory of F -zips developed in [MW04], [PWZ11] and [PWZ15] establishes the precise link between BT1's and group theory. Specifically, isomorphism classes of BT1's of height n and dimension d correspond bijectively to E -orbits in GL_n , where E is the zip group (see section 4.1). The stack of F -zips of type (n, d) can be defined as the quotient stack $F\text{-Zip}^{n,d} = [E \backslash GL_n]$. Moreover, there is a natural morphism of stacks $BT_1^{n,d} \rightarrow F\text{-Zip}^{n,d}$, where $BT_1^{n,d}$ is the stack of BT1's of height n and dimension d over k . More generally, let G be a connected reductive group over \mathbf{F}_p , and $P, Q \subset G$ parabolic subgroups. Let $L \subset P$ and $M \subset Q$ be Levi subgroups and assume that $\varphi(L) = M$, where $\varphi : G \rightarrow G$ is the Frobenius homomorphism. One can define the stack of G -zips of type $\mathcal{Z} = (G, P, L, Q, M, \varphi)$ as the quotient stack $G\text{-Zip}^{\mathcal{Z}} = [E \backslash G]$, where $E \subset P \times Q$ is the zip group (see section 1.1). For example, if G is the automorphism group of a PEL-datum, then $G\text{-Zip}^{\mathcal{Z}}(k)$ classifies BT1's over k of type \mathcal{Z} endowed with this additional structure. If W denotes the Weyl group of G , the E -orbits in G are parametrized by a subset ${}^I W \subset W$ (see section 1.3). Denote by $G_w \subset G$ the E -orbit corresponding to $w \in {}^I W$ and put $Z_w := [E \backslash G_w]$ the corresponding zip stratum.

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In [GK], we defined for each parabolic $P_0 \subset P$ the stack of partial zip flags $G\text{-ZipFlag}^{(\mathcal{Z}, P_0)}$ endowed with a natural projection $\pi : G\text{-ZipFlag}^{(\mathcal{Z}, P_0)} \rightarrow G\text{-Zip}^{\mathcal{Z}}$ which makes it a P/P_0 -bundle over $G\text{-Zip}^{\mathcal{Z}}$. This defines a tower of stacks above $G\text{-Zip}^{\mathcal{Z}}$. Moreover, the stack $G\text{-ZipFlag}^{(\mathcal{Z}, P_0)}$ admits two natural stratifications. In general, one is finer than the other, but they coincide when P_0 is a Borel subgroup. The fine strata $Z_{P_0, w}$ are parametrized by $w \in {}^I W$, where ${}^I W \subset W$ is a subset containing ${}^I W$ (see section 2.1). The strata $Z_{P_0, w}$ attached to elements $w \in {}^I W$ are called minimal and satisfy $\pi(Z_{P_0, w}) = Z_w$ and the restriction $\pi : Z_{P_0, w} \rightarrow Z_w$ is finite. When $P_0 = P$, the stack $G\text{-ZipFlag}^{(\mathcal{Z}, P)}$ coincides with $G\text{-Zip}^{\mathcal{Z}}$ and the fine stratification is the stratification by E -orbits, whereas the coarse stratification is given by $P \times Q$ -orbits. In general, we say that a stratum $Z_{P_0, w}$ has coarse closure if it is open in the coarse stratum containing it. In particular, if $Z_{P_0, w}$ has coarse closure, its Zariski closure $\overline{Z}_{P_0, w}$ is normal.

In the formalism of G -zips, one can attach to each $w \in {}^I W$ a parabolic subgroup $P_w \subset P$. In the case $G = GL_n$, if H is a BT1 corresponding to $w \in {}^I W$ under the correspondence between BT1's and E -orbits, then P_w is the parabolic $P(H)$ defined above. This is proved in Prop. 4.3.1. Since P_w is canonically attached to w , it is natural to ask what special property is satisfied by the stratum $Z_{P_w, w}$ of $G\text{-ZipFlag}^{(\mathcal{Z}, P_w)}$. Our main theorem answers this question:

Theorem 1. [Th. 3.1.3] *Let $w \in {}^I W$. The following properties hold:*

- (i) $\pi : Z_{P_w, w} \rightarrow Z_w$ is an isomorphism.
- (ii) $Z_{P_w, w}$ has coarse closure.

Furthermore, among all parabolic subgroups P_0 such that ${}^z B \subset P_0 \subset P$, the canonical parabolic P_w is the smallest parabolic satisfying (i) and the largest parabolic satisfying (ii).

The Borel subgroup ${}^z B$ of the above theorem is defined in section 1.2. Note that property (i) is obviously satisfied for $P_0 = P$ and property (ii) is satisfied for $P_0 = {}^z B$ because fine and coarse strata coincide in this case. Hence the canonical parabolic P_w is the unique intermediate parabolic such that both properties are satisfied. As a consequence, the normalization of the Zariski closure \overline{Z}_w is the closed fine stratum $\overline{Z}_{P_w, w}$ in $G\text{-ZipFlag}^{(P_w, w)}$ (see Corollary 3.3.2).

Let X be the special fiber of a good reduction Hodge-type Shimura variety, and let G be the attached reductive \mathbf{F}_p -group (see section 3.4). In [Zha], Zhang has constructed a smooth map of stacks

$$\zeta : X \longrightarrow G\text{-Zip}^{\mathcal{Z}}$$

where \mathcal{Z} is the zip datum attached to X as in [GK] §6.2. The Ekedahl-Oort stratification of X is defined as the fibers of ζ . For $w \in {}^I W$, set $X_w := \zeta^{-1}(Z_w)$. For any parabolic ${}^z B \subset P_0 \subset P$, define the partial flag space X_{P_0} as the fiber product

$$\begin{array}{ccc} X_{P_0} & \xrightarrow{\zeta_{P_0}} & G\text{-ZipFlag}^{(\mathcal{Z}, P_0)} \\ \pi \downarrow & & \downarrow \pi_{P_0} \\ X & \xrightarrow{\zeta} & G\text{-Zip}^{\mathcal{Z}} \end{array}$$

For $w \in {}^I W$, define the fine stratum $X_{P_0, w} := \zeta_{P_0}^{-1}(Z_{P_0, w})$ of X_{P_0} . The space X_{P_0} is a generalization of the flag space considered by Ekedahl and Van der Geer

in [EvdG09], where they consider flags refining the Hodge filtration of an abelian variety.

Corollary 1. *[Cor. 3.4.1] Let $w \in {}^I W$ and write $w = xw_J$ as in (1.2.1). Then the map $\pi : \overline{X}_{P_w, w} \rightarrow \overline{X}_w$ is the normalization of \overline{X}_w and induces an isomorphism $X_{P_w, w} \simeq X_w$.*

For Siegel-type Shimura varieties, an analogous result to Cor. 1 was proved by Boxer in [Box15, Th.5.3.1] using different methods.

We now give an overview of the paper. In section 1, we review the theory of G -zips and prove a result on point stabilizers for later use. Section 2 is devoted to the stack of partial G -zips and its stratifications. We define minimal strata and give an explicit form for the restriction of the map π to a minimal stratum (Prop. 2.2.1). In section 3, we define the notion of special parabolic and explain the relevance of this notion with respect to the normalization of a closed stratum of $G\text{-Zip}^Z$. We prove Theorem 3.1.3 after giving criteria for properties (i) and (ii) above. Finally, we explain in section 4 the correspondence between the classical theory of BT1's and the theory of G -zips, following [PWZ11]. We establish the link between the parabolic P_w and the canonical parabolic of a BT1.

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1. REVIEW OF G -ZIPS

We will need to review some facts about the stack of G -zips found in [PWZ11] and prove a result on the stabilizer of an element by the group E .

1.1. The stack $G\text{-Zip}^Z$. We fix an algebraic closure k of \mathbf{F}_p . A zip datum is a tuple $Z = (G, P, L, Q, M, \varphi)$ where G is a connected reductive group over \mathbf{F}_p , $\varphi : G \rightarrow G$ is the Frobenius homomorphism, $P, Q \subset G$ are parabolic subgroups of G_k , $L \subset P$ and $M \subset Q$ are Levi subgroups of P and Q respectively. One imposes the condition $\varphi(L) = M$. One can attach to Z a zip group E defined by

$$(1.1.1) \quad E := \{(p, q) \in P \times Q, \varphi(\overline{p}) = \overline{q}\}$$

where $\overline{p} \in L$ and $\overline{q} \in M$ denote the projections of p and q via the isomorphisms $P/R_u(P) \simeq L$ and $Q/R_u(Q) \simeq M$. We let $G \times G$ act on G via $(a, b) \cdot g := agb^{-1}$ and we obtain by restriction an action of E on G . The stack of G -zips is then isomorphic to the following quotient stack:

$$(1.1.2) \quad G\text{-Zip}^Z \simeq [E \backslash G].$$

When we want to specify the zip datum Z , we write sometimes E_Z for the zip group E .

1.2. Frame. A frame for Z is a triple (B, T, z) where (B, T) is a Borel pair and $z \in G(k)$ satisfying the following conditions:

- (i) $B \subset Q$
- (ii) ${}^z T \subset L$
- (iii) ${}^z B \subset P$
- (iv) $\varphi({}^z B \cap L) = B \cap M$
- (v) $\varphi({}^z T) = T$.

We fix throughout a frame (B, T, z) and we define:

- (1) $\Phi \subset X^*(T)$: the set of T -roots of G .
- (2) Φ_+ : the set of positive roots with respect to B .
- (3) $\Delta \subset \Phi_+$: the set of positive simple roots.
- (4) For $\alpha \in \Phi$, let $s_\alpha \in W$ be the corresponding reflection. Then $(W, \{s_\alpha\}_{\alpha \in \Delta})$ is a Coxeter group and we denote by $\ell : W \rightarrow \mathbf{N}$ the length function.
- (5) For $K \subset \Delta$, Let $W_K \subset W$ be the subgroup generated by the s_α for $\alpha \in K$. Let $w_0 \in W$ be the longest element and $w_{0,K}$ the longest element in W_K .
- (6) If $D \subset G$ is a Levi subgroup containing T , its type is the subset $K \subset \Delta$ such that $W(D) = W_K$. If R is a parabolic containing B , the type of R is the type of the Levi subgroup of R containing T . The type of an arbitrary parabolic R is the type of its unique conjugate containing B . Let $I \subset \Delta$ (resp. $J \subset \Delta$) be the type of P (resp. Q).
- (7) For $K \subset \Delta$, ${}^K W$ (resp. W^K) : the subset of elements $w \in W$ which are minimal in the coset $W_K w$ (resp. $w W_K$).
- (8) For $K, R \subset \Delta$, ${}^K W^R := {}^K W \cap W^R$.
- (9) For an element $x \in {}^I W^J$, define $I_x := J \cap x^{-1} I$. By Proposition 2.7 in [PWZ11], any element $w \in W_I x W_J$ can be uniquely written as

$$(1.2.1) \quad w = x w_J, \quad \text{with } w_J \in {}^{I_x} W_J.$$

For $w \in W$, one has an equivalence:

$$(1.2.2) \quad w \in {}^I W \iff {}^z B \cap L = {}^{zw} B \cap L.$$

1.3. Stratification. For $w \in W$, choose a representative $\dot{w} \in N_G(T)$, such that $(w_1 w_2)^\cdot = \dot{w}_1 \dot{w}_2$ whenever $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$ (this is possible by choosing a Chevalley system, see [ABD⁺66], Exp. XXIII, §6). For $h \in G(k)$, denote by $\mathcal{O}_{\mathcal{Z}}(h)$ the E -orbit of h in G and define $\mathfrak{o}_{\mathcal{Z}}(h) := [E \setminus \mathcal{O}_{\mathcal{Z}}(h)]$. By Theorem 7.5 in [PWZ11], there is a bijection:

$$(1.3.1) \quad {}^I W \rightarrow \{E\text{-orbits in } G\}, \quad w \mapsto G_w := \mathcal{O}(z \dot{w}).$$

Furthermore, for all $w \in {}^I W$, one has:

$$(1.3.2) \quad \dim(G_w) = \ell(w) + \dim(P).$$

For $w \in {}^I W$, we endow the locally closed subset G_w with the reduced structure, and we define the corresponding zip stratum of $G\text{-Zip}^{\mathcal{Z}}$ by $Z_w := [E \setminus G_w]$.

1.4. Point stabilizers.

Definition 1.4.1 ([PWZ11, Def. 5.1]). *Let $w \in {}^I W$. There is a largest subgroup M_w of ${}^{w^{-1}z^{-1}} L$ satisfying $\varphi({}^{zw} M_w) = M_w$.*

In *loc. cit.* §5.1, this subgroup is denoted by H_w . It is a Levi subgroup of G contained in M . We define also:

$$(1.4.1) \quad L_w := {}^{zw} M_w \subset L$$

$$(1.4.2) \quad P_w := L_w {}^z B \subset P$$

$$(1.4.3) \quad Q_w := M_w B \subset Q$$

Since $\varphi(L_w) = M_w$, we obtain a zip datum $\mathcal{Z}_w := (G, P_w, L_w, Q_w, M_w, \varphi)$. Note that (B, T, z) is again a frame for \mathcal{Z}_w . If an algebraic group G acts on a k -scheme X and $x \in X(k)$, we denote by $\text{Stab}_G(x)$ the scheme-theoretical stabilizer of x . For

an algebraic group H , we denote by H_{red} the underlying reduced algebraic group and by H° the identity component of H .

Lemma 1.4.2.

- (1) One has $\text{Stab}_E(z\dot{w})_{\text{red}} = A \ltimes R$ where $A \subset L_w \times M_w$ is the finite group
- $$(1.4.4) \quad A := \{(x, \varphi(x)), x \in L_w, {}^{z\dot{w}}\varphi(x) = x\}$$
- and R is a unipotent smooth connected normal subgroup.
- (2) One has $\text{Stab}_E(z\dot{w})^\circ \subset {}^z B \times B$.

Proof. The first part is Th. 8.1 in *loc. cit.* To prove (2), it suffices to show that $\text{Stab}_E(z\dot{w})^\circ \subset {}^z B \times G$, or equivalently $\text{Lie}(\text{Stab}_E(z\dot{w})) \subset \text{Lie}({}^z B) \times \text{Lie}(G)$. We follow the proof of Th. 8.5 of *loc. cit.* An arbitrary tangent vector of E at 1 has the form $(1 + dp, 1 + dv)$ for $dp \in \text{Lie}(P)$ and $dv \in \text{Lie}(V)$. This element stabilizes $z\dot{w}$ if and only if $dp = \text{Ad}_{z\dot{w}}(dv)$. Hence

$$(1.4.5) \quad dp \in \text{Lie}(P) \cap \text{Ad}_{z\dot{w}}(\text{Lie}(V)) = \text{Lie}(P \cap {}^{zw}V)$$

Hence it suffices to show $P \cap {}^{zw}V \subset {}^z B$. This amounts to $L \cap {}^{zw}V \subset L \cap {}^z B$ and equivalently $M \cap \varphi({}^{zw}V) \subset M \cap \varphi({}^z B) = M \cap B$. This is proved in Prop. 4.12 of *loc. cit.* More precisely, the authors define in construction 4.3 a group V_x (note that the element z is denoted by g there), where $w = xw_J$ is a decomposition as in (1.2.1). One has $V_x = M \cap \varphi({}^{z\dot{x}}V) = M \cap \varphi({}^{zw}V)$ because $w_J \in W_J$, so ${}^{w_J}V = V$. Prop. 4.12 of *loc. cit.* shows that $(M \cap B, T, 1)$ is a frame for $\mathcal{Z}_{\dot{x}}$, so in particular one has $V_x \subset M \cap B$. This terminates the proof of the lemma. \square

2. THE STACK OF PARTIAL ZIP FLAGS

We recall in this section some of the results of [GK].

2.1. Fine and coarse flag strata. For each parabolic subgroup P_0 satisfying ${}^z B \subset P_0 \subset P$, we defined in *loc. cit.* §2 a stack $G\text{-ZipFlag}^{(\mathcal{Z}, P_0)}$ which parametrizes G -zips of type \mathcal{Z} endowed with a compatible P_0 -torsor. There is an isomorphism

$$(2.1.1) \quad G\text{-ZipFlag}^{(\mathcal{Z}, P_0)} \simeq [E \backslash (G \times P/P_0)]$$

where E acts on $G \times P/P_0$ by $(a, b) \cdot (g, xP_0) := (agb^{-1}, axP_0)$. Furthermore, there is a natural projection map $\pi : G\text{-ZipFlag}^{(\mathcal{Z}, P_0)} \rightarrow G\text{-Zip}^{\mathcal{Z}}$ which is a P/P_0 -bundle.

Denote by $L_0 \subset P_0$ the Levi subgroup containing ${}^z T$ (note that $L_0 \subset L$). We define a second zip datum $\mathcal{Z}_0 = (G, P_0, L_0, Q_0, M_0, \varphi)$ by setting:

$$(2.1.2) \quad M_0 := \varphi(L_0) \subset M$$

$$(2.1.3) \quad Q_0 := M_0 B \subset Q.$$

Note that (B, T, z) is again a frame of \mathcal{Z}_0 . By *loc. cit.* §3.1, there is a natural morphism of stacks

$$(2.1.4) \quad \Psi_{P_0} : G\text{-ZipFlag}^{(\mathcal{Z}, P_0)} \rightarrow G\text{-Zip}^{\mathcal{Z}_0}$$

which is an \mathbf{A}^r -bundle for $r = \dim(P/P_0)$ (Proposition 3.1.1 in *loc. cit.*). It is induced by the map $G \times P \rightarrow G$, $(g, a) \mapsto a^{-1}g\varphi(\overline{a})$. Let I_0 and J_0 denote respectively the types of P_0 and Q_0 . For $w \in {}^{I_0}W$, we define the fine flag stratum $Z_{P_0, w}$ of $G\text{-ZipFlag}^{(\mathcal{Z}, P_0)}$ as the locally closed substack

$$(2.1.5) \quad Z_{P_0, w} := \Psi_{P_0}^{-1}(\mathfrak{o}_{\mathcal{Z}_0}(z\dot{w}))$$

endowed with the reduced structure. Explicitly, one has $Z_{P_0,w} = [E \backslash G_{P_0,w}]$ where $G_{P_0,w}$ is the algebraic subvariety of $G \times P/P_0$ defined by

$$(2.1.6) \quad G_{P_0,w} := \{(g, aP_0) \in G \times P/P_0, a^{-1}g\varphi(\bar{a}) \in \mathcal{O}_{\mathcal{Z}_0}(z\dot{w})\}.$$

Denote by $\mathbf{Brh}^{\mathcal{Z}_0}$ the quotient stack $[P_0 \backslash G/Q_0]$, called the Bruhat stack. Since $E_{\mathcal{Z}_0} \subset P_0 \times Q_0$, there is a natural projection morphism $\beta : G\text{-Zip}^{\mathcal{Z}_0} \rightarrow \mathbf{Brh}^{\mathcal{Z}_0}$. The composition $\Psi_{P_0} \circ \beta$ gives a smooth map of stacks

$$(2.1.7) \quad \psi_{P_0} : G\text{-ZipFlag}^{(\mathcal{Z}, P_0)} \rightarrow \mathbf{Brh}^{\mathcal{Z}_0}.$$

By [Wed, Lem. 1.4], the set $\{z\dot{x}, x \in {}^{I_0}W^{J_0}\}$ is a set of representatives of the $P_0 \times Q_0$ -orbits in G (pay attention to the fact that ${}^zB \subset P$). For $x \in {}^{I_0}W^{J_0}$, write $\mathfrak{b}(x) := [P_0 \backslash (P_0 z\dot{x}Q_0)/Q_0]$ (locally closed substack of $\mathbf{Brh}^{\mathcal{Z}_0}$) and define the coarse flag stratum $\mathbf{Z}_{P_0,x}$ as

$$(2.1.8) \quad \mathbf{Z}_{P_0,x} := \psi_{P_0}^{-1}(\mathfrak{b}(x))$$

endowed with the reduced structure. Explicitly, one has $\mathbf{Z}_{P_0,x} = [E \backslash \mathbf{G}_{P_0,x}]$ where $\mathbf{G}_{P_0,x}$ is the subvariety of $G \times P/P_0$ defined by

$$(2.1.9) \quad \mathbf{G}_{P_0,x} := \{(g, aP_0) \in G \times P/P_0, ag\varphi(\bar{a})^{-1} \in P_0 z\dot{x}Q_0\}.$$

All fine and coarse flag strata are smooth. A coarse stratum is a union of fine strata and the Zariski closure of a coarse flag stratum is normal. In each coarse stratum there is a unique open fine stratum.

Definition 2.1.1. *We say that a fine flag stratum Z has coarse closure if it is open in the coarse stratum that contains it, equivalently, if its Zariski closure coincides with the Zariski closure of a coarse flag stratum.*

In particular, the Zariski closure \overline{Z} of such a stratum is normal.

2.2. Minimal strata. Recall that we defined in [GK] a minimal flag stratum as a flag stratum $Z_{P_0,w}$ parametrized by an element $w \in {}^I W$. For a minimal stratum one has $\pi(Z_{P_0,w}) = Z_w$ and the induced morphism $\pi : Z_{P_0,w} \rightarrow Z_w$ is finite (Prop. 3.2.2 of *loc. cit.*). The following proposition shows that it is also étale. For $w \in {}^I W$, denote by $\tilde{\pi} : G_{P_0,w} \rightarrow G_w$ the first projection, it is an E -equivariant map.

Proposition 2.2.1. *Let $w \in {}^I W$ and denote by $S := \text{Stab}_E(z\dot{w})$ the stabilizer of $z\dot{w}$ in E and define $S_{P_0} := S \cap (P_0 \times G)$.*

(1) *There is a commutative diagram*

$$\begin{array}{ccc} Z_{P_0,w} & \xrightarrow{\simeq} & [1/S_{P_0}] \\ \pi \downarrow & & \downarrow \\ Z_w & \xrightarrow{\simeq} & [1/S] \end{array}$$

where the horizontal maps are isomorphisms and the right-hand side vertical map is the natural projection.

(2) *The map $\pi : Z_{P_0,w} \rightarrow Z_w$ is finite étale.*

(3) *The map $\pi : Z_{P_0,w} \rightarrow Z_w$ is an isomorphism if and only if the inclusion $S \subset P_0 \times G$ holds.*

Proof. We first prove (1). There is a natural identification $G_w \simeq [E/S]$ because G_w is the E -orbit of $z\dot{w}$. It follows that $Z_w \simeq [E \backslash E/S] \simeq [1/S]$. Similarly, we claim that the variety $G_{P_0, w}$ consists of a single E -orbit. This was proved in [GK15] Proposition 5.4.5 in the case when P_0 is a Borel subgroup. For a general P_0 , we may reduce to the Borel case as follows: By Proposition 3.2.2 of [GK], we have a natural E -equivariant surjective projection map $G^{zB, w} \rightarrow G_{P_0, w}$, hence $G_{P_0, w}$ consists of a single E -orbit. We thus can identify $Z_{P_0, w} \simeq [E \backslash E/S']$ where $S' = \text{Stab}_E(z\dot{w}, 1)$. It is clear that $S' = S_{P_0}$, so the result follows.

We now show (2). By Prop. 3.2.2 of *loc. cit.*, we know that $\pi : Z_{P_0, w} \rightarrow Z_w$ is finite. By (1), it is equivalent to show that S/S_{P_0} is an étale scheme. By Lemma 1.4.2 (2), we have $S^\circ \subset S \cap ({}^zB \times G) \subset S_{P_0}$. Hence the quotient map $S \rightarrow S/S_{P_0}$ factors through a surjective map $\pi_0(S) \rightarrow S/S_{P_0}$, which shows that S/S_{P_0} is étale.

Finally, the last assertion follows immediately from (1). \square

3. THE CANONICAL PARABOLIC

We fix an element $w \in {}^I W$. Recall that we defined in (1.4.3) a parabolic subgroup $P_w \subset P$ that we will call the canonical parabolic subgroup attached to w .

3.1. Special parabolic subgroup. Let P_0 be a parabolic subgroup of G such that ${}^zB \subset P_0 \subset P$.

Definition 3.1.1. *We say that P_0 is a special parabolic subgroup for w if the following properties are satisfied:*

- (i) *The map $\pi : Z_{P_0, w} \rightarrow Z_w$ is an isomorphism.*
- (ii) *The stratum $Z_{P_0, w}$ has coarse closure.*

Using the notations of Proposition 2.2.1, property (i) is equivalent to $S \subset P_0 \times G$. The justification of this definition is the following: For $P_0 = P$, Condition (i) is obviously satisfied. On the other hand, if $P_0 = {}^zB$, then (ii) is satisfied because coarse and fine strata coincide. For a given w , a special parabolic for w is an intermediate parabolic subgroup P_0 satisfying both conditions. A priori neither the existence nor the uniqueness of such a parabolic is clear.

We give justification for this definition. Let P_0 be a special parabolic subgroup for w . We have morphisms:

$$(3.1.1) \quad \pi : \overline{Z}_{P_0, w} \rightarrow \overline{Z}_w, \quad \tilde{\pi} : \overline{G}_{P_0, w} \rightarrow \overline{G}_w$$

which yield isomorphisms $Z_{P_0, w} \simeq Z_w$ and $G_{P_0, w} \simeq G_w$. Since $Z_{P_0, w}$ has coarse closure, the stack (resp. variety) $\overline{Z}_{P_0, w}$ (resp. $\overline{G}_{P_0, w}$) is normal. We deduce the following proposition:

Proposition 3.1.2. *Let P_0 be a special parabolic subgroup for $w \in {}^I W$. Write $w = xw_J$ as in (1.2.1). Then the normalization of the Zariski closure of G_w is the variety:*

$$(3.1.2) \quad \overline{G}_{P_0, w} = \overline{G}_{P_0, x} = \{(g, aP_0) \in G \times P/P_0, a^{-1}g\varphi(\bar{a}) \in \overline{P_0 z\dot{w}Q_0}\}$$

and the first projection induces an isomorphism $G_{P_0, w} \simeq G_w$.

The main result of this paper is the following theorem, whose proof will follow from the results of sections 3.2 and 3.3.

Theorem 3.1.3. *Let $w \in {}^I W$. The canonical parabolic P_w is the unique special parabolic for w . More precisely, among all parabolic subgroups ${}^z B \subset P_0 \subset P$, the following holds:*

- (a) P_w is the smallest parabolic P_0 such that $\pi : Z_{P_0, w} \rightarrow Z_w$ is an isomorphism.
- (b) P_w is the largest parabolic P_0 such that $Z_{P_0, w}$ has coarse closure.

3.2. A criterion for Condition (i).

Lemma 3.2.1. *Let ${}^z B \subset P_0 \subset P$ be a parabolic subgroup. The following assertions are equivalent:*

- (1) *The map $\pi : Z_{P_0, w} \rightarrow Z_w$ is an isomorphism.*
- (2) *One has $P_w \subset P_0$.*

Proof. Using the notation of Proposition 2.2.1, we know that $\pi : Z_{P_0, w} \rightarrow Z_w$ is an isomorphism if and only if $S_{P_0} := S \cap (P_0 \times G) = S$. By the same proposition, we know that the quotient S/S_{P_0} is a finite affine étale scheme over k . In particular, we have $S \subset P_0 \times G$ if and only if $S_{\text{red}} \subset P_0 \times G$.

By Lemma 1.4.2, we can write $S_{\text{red}} = A \ltimes R$ with A the finite group given by equation (1.4.4) of Lemma 1.4.2 and R a smooth unipotent connected normal subgroup. Write $R_{P_0} := R \cap (P_0 \times G)$. The inclusion $R \subset S$ induces a closed embedding

$$(3.2.1) \quad R/R_{P_0} \rightarrow S/S_{P_0}.$$

Hence R/R_{P_0} is a finite, smooth, connected, affine k -scheme, so $R/R_{P_0} = \text{Spec}(k)$, hence $R \subset P_0 \times G$. It follows that $S \subset P_0 \times G$ if and only if $A \subset P_0 \times G$, which is equivalent to $A_1 \subset P_0$, where

$$(3.2.2) \quad A_1 := \{x \in L_w, {}^{z\dot{w}}\varphi(x) = x\}.$$

By Steinberg's theorem we can write ${}^{z\dot{w}}\varphi(a) = a^{-1}\varphi(a)$ with $a \in G(k)$. Then it is easy to see that the subgroup ${}^a L_w$ is defined over \mathbf{F}_p and the inclusion (3.2.2) is equivalent to

$$(3.2.3) \quad ({}^a L_w)(\mathbf{F}_p) \subset {}^a P_0.$$

Note that both ${}^a L_w$ and ${}^a P_0$ contain the torus ${}^{az}T$, which is defined over \mathbf{F}_p . Thus Lemma 3.2.2 below shows that (3.2.3) is equivalent to ${}^a L_w \subset {}^a P_0$, hence $L_w \subset P_0$, which is the same as $P_w \subset P_0$. This terminates the proof. \square

Lemma 3.2.2. *Let G be a connected reductive group over \mathbf{F}_p . Let L be a Levi subgroup of G defined over \mathbf{F}_p and P be a parabolic subgroup of G_k such that there exists a maximal \mathbf{F}_p -torus T contained in L and P . Assume further that*

$$(3.2.4) \quad L(\mathbf{F}_p) \subset P.$$

Then one has $L \subset P$.

Proof. We want to thank Wushi Goldring for suggesting this proof to us. We first reduce to the case when P is defined over \mathbf{F}_p . The inclusion $L(\mathbf{F}_p) \subset P$ is equivalent to $L(\mathbf{F}_p) \subset P' := \bigcap_{r \in \mathbf{N}} \varphi^r(P)$ and P' is defined over \mathbf{F}_p . Hence we may assume that P is defined over \mathbf{F}_p . Consider the groups $L \cap P \subset L$. Then $(L \cap P)(\mathbf{F}_p) = L(\mathbf{F}_p)$. Hence it suffices to show the following claim: Let G be a connected reductive group over \mathbf{F}_p and $P \subset G$ be a parabolic subgroup defined over \mathbf{F}_p . Assume $P(\mathbf{F}_p) = G(\mathbf{F}_p)$. Then $P = G$.

To prove the claim, consider the flag variety $X := G/P$. It follows from Steinberg's theorem that $H^1(\mathbf{F}_p, P) = 1$, which implies that $X(\mathbf{F}_p) = G(\mathbf{F}_p)/P(\mathbf{F}_p) = \{1\}$. In the vein of Weil's conjectures, it is proved in [BP10] that there exists a polynomial $P(t) \in \mathbf{Z}[t]$ such that $|X(\mathbf{F}_{p^d})| = P_X(p^d)$ for all $d \geq 1$ and that P factors into a product $P_X(t) = (t-1)^r Q_X(t)$ where Q has non-negative coefficients. It follows that $P_X(t) = 1$, which implies $X = \text{Spec}(k)$, so $P = G$ and the claim is proved. This terminates the proof of the Lemma. \square

3.3. A criterion for Condition (ii). We examine Condition (ii) of Definition 3.1.1. Let ${}^z B \subset P_0 \subset P$ be a parabolic subgroup and let L_0, M_0, Q_0, Z_0 as defined in section 2.1.

Lemma 3.3.1. *Let ${}^z B \subset P_0 \subset P$ be a parabolic subgroup. The following assertions are equivalent:*

- (1) $Z_{P_0, w}$ has coarse closure.
- (2) One has ${}^{z\dot{w}} M_0 = L_0$.

Proof. The stratum $Z_{P_0, w}$ has coarse closure if and only if $\mathcal{O}_{E_{Z_0}}(z\dot{w}) \subset P_0 z\dot{w} Q_0$ is an open embedding, which is equivalent to the equality of their dimensions. Note that (B, T, z) is again a frame of Z_0 , so formula (1.3.2) shows that

$$(3.3.1) \quad \dim(\mathcal{O}_{E_{Z_0}}(z\dot{w})) = \dim(P_0) + \ell(w).$$

On the other hand, we have:

$$(3.3.2) \quad \dim(P_0 z\dot{w} Q_0) = 2 \dim(P_0) - \dim(\text{Stab}_{P_0 \times Q_0}(z\dot{w})).$$

The stabilizer $\text{Stab}_{P_0 \times Q_0}(z\dot{w})$ is the subgroup:

$$\begin{aligned} \text{Stab}_{P_0 \times Q_0}(z\dot{w}) &= \{(a, b) \in P_0 \times Q_0, az\dot{w} = z\dot{w}b\} \\ &\simeq \{a \in P_0, (z\dot{w})^{-1}az\dot{w} \in Q_0\} \\ &= P_0 \cap {}^{z\dot{w}} Q_0. \end{aligned}$$

Hence $Z_{P_0, w}$ has coarse closure if and only if $\dim(P_0/(P_0 \cap {}^{z\dot{w}} Q_0)) = \ell(w)$. Since the property is satisfied when $P_0 = {}^z B$, we have $\dim({}^z B/({}^z B \cap {}^{z\dot{w}} B)) = \ell(w)$, so we can rewrite the property as

$$(3.3.3) \quad \dim((P_0 \cap {}^{z\dot{w}} Q_0)/({}^z B \cap {}^{z\dot{w}} B)) = \dim(P_0/{}^z B)$$

Since (B, T, z) is a frame for Z_0 and ${}^I W \subset {}^{I_0} W$, equation (1.2.2) shows that $P_0 \cap {}^z B = P_0 \cap {}^{z\dot{w}} B$, thus the inclusion $P_0 \cap {}^{z\dot{w}} Q_0 \subset P_0$ induces an embedding

$$(3.3.4) \quad (P_0 \cap {}^{z\dot{w}} Q_0)/({}^z B \cap {}^{z\dot{w}} B) \rightarrow P_0/{}^z B.$$

Hence (3.3.3) is satisfied if and only if the image of $P_0 \cap {}^{z\dot{w}} Q_0$ is open in $P_0/{}^z B$. Since $P_0/{}^z B \simeq L_0/({}^z B \cap L_0)$ it is also equivalent to $L_0 \cap {}^{z\dot{w}} Q_0$ having open image in $L_0/({}^z B \cap L_0)$.

Denote by B' the opposite Borel in G of B with respect to T . Then ${}^z B' \cap L_0$ is the opposite Borel of ${}^z B \cap L_0$ in L_0 with respect to ${}^z T$. Thus the image of $L_0 \cap {}^{z\dot{w}} Q_0$ is open in $L_0/({}^z B \cap L_0)$ if and only if ${}^z B' \cap L_0 \subset {}^{z\dot{w}} Q_0$. It follows immediately from equation (1.2.2) that ${}^z B' \cap L_0 = {}^{z\dot{w}} B' \cap L_0$. Finally, we find that $Z_{P_0, w}$ has coarse closure if and only if

$$(3.3.5) \quad B' \cap (z\dot{w})^{-1} L_0 \subset Q_0.$$

The groups $B' \cap (z\dot{w})^{-1}L_0$ and $B \cap (z\dot{w})^{-1}L_0$ are opposite Borel subgroups of $(z\dot{w})^{-1}L_0$ containing T . Since $B \subset Q_0$, equation (3.3.5) is simply equivalent to $(z\dot{w})^{-1}L_0 \subset Q_0$, which is equivalent to $(z\dot{w})^{-1}L_0 = M_0$. This terminates the proof. \square

Proof of Theorem 3.1.3. The result follows immediately by combining Lemmas 3.2.1 and 3.3.1. \square

Corollary 3.3.2. *Write $w = xw_J$ as in (1.2.1). The normalization of the Zariski closure of G_w is the variety:*

$$(3.3.6) \quad \overline{G}_{P_w, w} = \overline{\mathbf{G}}_{P_w, x} = \{(g, aP_w) \in G \times P/P_w, a^{-1}g\varphi(\bar{a}) \in \overline{P_w z\dot{w}Q_w}\}$$

and the first projection induces an isomorphism $G_{P_w, w} \simeq G_w$.

3.4. Shimura varieties and Ekedahl-Oort strata. Let X be the special fiber of a Hodge-type Shimura variety attached to a Shimura datum (\mathbf{G}, \mathbf{X}) with hyperspecial level at p . Write $G := G_{\mathbf{Z}_p} \times \mathbf{F}_p$, where $G_{\mathbf{Z}_p}$ is a reductive \mathbf{Z}_p -model of $\mathbf{G}_{\mathbf{Q}_p}$. By Zhang [Zha], there exists a smooth morphism of stacks

$$(3.4.1) \quad \zeta : X \longrightarrow G\text{-Zip}^{\mathcal{Z}}$$

where \mathcal{Z} is the zip datum attached to (\mathbf{G}, \mathbf{X}) as in [GK] §6.2. The Ekedahl-Oort stratification of X is defined as the fibers of ζ . For $w \in {}^I W$, set:

$$(3.4.2) \quad X_w := \zeta^{-1}(Z_w).$$

By the smoothness of ζ , this defines a stratification of X . Let ${}^z B \subset P_0 \subset P$ be a parabolic subgroup and define the partial flag space X_{P_0} as the fiber product

$$(3.4.3) \quad \begin{array}{ccc} X_{P_0} & \xrightarrow{\zeta_{P_0}} & G\text{-ZipFlag}^{(\mathcal{Z}, P_0)} \\ \pi \downarrow & & \downarrow \pi_{P_0} \\ X & \xrightarrow{\zeta} & G\text{-Zip}^{\mathcal{Z}} \end{array}$$

The map $\pi : X_{P_0} \rightarrow X$ is a P/P_0 -bundle. For $w \in {}^{I_0} W$ and $x \in {}^{I_0} W^{J_0}$ define

$$(3.4.4) \quad X_{P_0, w} := \zeta_0^{-1}(Z_{P_0, w})$$

$$(3.4.5) \quad \mathbf{X}_{P_0, x} := \zeta_0^{-1}(\mathbf{Z}_{P_0, x}).$$

We call $X_{P_0, w}$ the fine stratum attached to $w \in {}^{I_0} W$ and $\mathbf{X}_{P_0, x}$ the coarse stratum attached to $x \in {}^{I_0} W^{J_0}$. All coarse and fine strata are smooth and locally closed, they define stratifications of X_{P_0} , and the Zariski closure of a coarse stratum is normal.

Corollary 3.4.1. *Let $w \in {}^I W$ and write $w = xw_J$ as in (1.2.1). The morphism*

$$(3.4.6) \quad \pi : \overline{X}_{P_w, w} = \overline{\mathbf{X}}_{P_w, x} \longrightarrow \overline{X}_w$$

is the normalization of \overline{X}_w and it induces an isomorphism $X_{P_w, w} \simeq X_w$.

4. THE CANONICAL FILTRATION

Most of the content of this section can be found in [PWZ11]. We merely unwind their proofs to make the link between the canonical filtration of a Dieudonné space and the group L_w defined previously. See also [Moo01, §4.4] and [Box15] for related results.

4.1. Dieudonné spaces and GL_n -zips. Let H be a truncated Barsotti-Tate groups of level 1 over k of height n . Set $d := \dim(\text{Lie}(H))$ and write $D := \mathbf{D}(H)$ for its Dieudonné space. It is a k -vector space of dimension n endowed with a σ -linear endomorphism $F : D \rightarrow D$, a σ^{-1} -linear endomorphism $V : D \rightarrow D$ satisfying the conditions:

- (1) $\text{Ker}(F) = \text{Im}(V)$,
- (2) $\text{Ker}(V) = \text{Im}(F)$,
- (3) $\text{rk}(V) = d$.

We say that (D, F, V) is a Dieudonné space of height n and dimension d . Let $M_n^{(r)}(k)$ be the set of matrices in $M_n(k)$ of rank r . After choosing a k -basis of D , we may write $F = a \otimes \sigma$ and $V = b \otimes \sigma^{-1}$, where (a, b) is in the set

$$\mathcal{X} := \{(a, b) \in M_n^{(n-d)}(k) \times M_n^{(d)}(k), a\sigma(b) = \sigma(b)a = 0\}.$$

Note that for $(a, b) \in \mathcal{X}$, we have $\text{Ker}(a) = \text{Im}(\sigma(b)) = \sigma(\text{Im}(b))$ and $\text{Im}(a) = \text{Ker}(\sigma(b)) = \sigma(\text{Ker}(b))$.

It is easy to see that two such pairs (a, b) and (a', b') yield isomorphic Dieudonné spaces if and only if there exists $M \in GL_n(k)$ such that

$$(4.1.1) \quad a' = Ma\sigma(M)^{-1} \quad \text{and} \quad b' = Mb\sigma^{-1}(M)^{-1}.$$

This defines an action of $GL_n(k)$ on \mathcal{X} and we obtain a bijection between isomorphism classes of Dieudonné spaces of height n and dimension d and the set of $GL_n(k)$ -orbits in \mathcal{X} .

Let (e_1, \dots, e_n) the canonical basis of k^n and define

$$(4.1.2) \quad V_1 := \text{Span}(e_1, \dots, e_{n-d})$$

$$(4.1.3) \quad V_2 := \text{Span}(e_{n-d+1}, \dots, e_n)$$

Define $P := \text{Stab}(V_2)$, $Q := \text{Stab}(V_1)$, $L := P \cap Q$, $U := R_u(P)$ and $V := R_u(Q)$. Consider the set

$$(4.1.4) \quad \mathcal{Y} := \{(a, b) \in \mathcal{X}, \text{Ker}(a) = V_2\}.$$

The action of $GL_n(k)$ on \mathcal{X} restricts to an action of $P(k)$ on \mathcal{Y} and the inclusion $\mathcal{Y} \subset \mathcal{X}$ induces a bijection between $P(k)$ -orbits in \mathcal{Y} and $GL_n(k)$ -orbits in \mathcal{X} .

Lemma 4.1.1. *There is a natural bijection*

$$(4.1.5) \quad \Psi : \mathcal{Y} \longrightarrow GL_n(k)/V$$

Proof. Let $(a, b) \in \mathcal{Y}$ and choose a subspace $H \subset k^n$ such that $\text{Im}(a) \oplus H = k^n$. Define a matrix $f_H \in GL_n(k)$ by the following diagram

$$(4.1.6) \quad \begin{array}{ccccc} k^n & = & V_1 & \oplus & V_2 \\ f_H \downarrow & & \downarrow a & & \downarrow \sigma(b)^{-1} \\ k^n & = & \text{Im}(a) & \oplus & H \end{array}.$$

In other words, $f_H v = av$ for $v \in V_1$, and if $v \in V_2$, then $f_H v$ is the only element $h \in H$ such that $\sigma(b)h = v$ (note that $\text{Im}(a) = \text{Ker}(\sigma(b))$, so this element is well-defined). It is clear that f_H is invertible.

If H' denotes another subspace such that $\text{Im}(a) \oplus H' = k^n$, then we can write $f_{H'} = f_H \alpha$, for some $\alpha \in GL_n(k)$. It is clear that $\alpha(v) = v$ for all $v \in V_1$. Furthermore, for $v \in V_2$, one must have $\sigma(b)(f_{H'} v - f_H v) = 0$, thus $f_H(\alpha(v) - v) \in$

$\text{Ker}(\sigma(b)) = \text{Im}(a)$, so $\alpha(v) - v \in V_1$. This shows that $\alpha \in V$. It follows that $(a, b) \mapsto f_H$ induces a well-defined map $\Psi : \mathcal{Y} \rightarrow GL_n(k)/V$. We leave it to the reader to check that this map is a bijection. \square

Define a subgroup of $P \times Q$ by

$$(4.1.7) \quad E := \{(M_1, M_2) \in P \times Q, \varphi(\overline{M}_1) = \overline{M}_2\}.$$

Let this group acts on $GL_n(k)$ by the rule $(M_1, M_2) \cdot g := M_1 g M_2^{-1}$.

Proposition 4.1.2. *The map Ψ induces a bijection*

$$(4.1.8) \quad P(k) \backslash \mathcal{Y} \rightarrow E \backslash GL_n(k)$$

Hence there is a bijection between isomorphism classes of Dieudonné spaces of height n and dimension d and the set of E -orbits in $GL_n(k)$.

Proof. Let $M \in P(k)$, $(a, b) \in \mathcal{Y}$ and set $(a', b') := (Ma\sigma(M)^{-1}, Mb\sigma^{-1}(M)^{-1})$. Note that $\text{Im}(a') = M(\text{Im}(a))$. Choose a subspace H such that $\text{Im}(a) \oplus H = k^n$ and set $H' := M(H)$. Let $\overline{M} \in L(k)$ denote the Levi component of $M \in P(k)$. Finally, write f_H and $f_{H'}$ for the maps attached to (a, b, H) and (a', b', H') respectively by the previous construction. We claim that one has the relation:

$$(4.1.9) \quad M f_H = f_{H'} \sigma(\overline{M}).$$

First assume $v \in V_1$. Then $f_{H'} \sigma(\overline{M})v = Ma\sigma(M)^{-1}\overline{M}v$. Since $\sigma(M)^{-1}\overline{M} \in U$, we have $\sigma(M)^{-1}\overline{M}v - v \in V_2$, hence $f_{H'} \sigma(\overline{M})v = Mav = M f_H v$.

Now if $v \in V_2$, the element $f_H v$ is the only element $h \in H$ satisfying $\sigma(b)h = v$. Similarly, $f_{H'} \sigma(\overline{M})v$ is the only element $h' \in H' = M(H)$ such that $\sigma(b')h' = \sigma(\overline{M})v$. Hence $\sigma(\overline{M})^{-1}\sigma(M)\sigma(b)M^{-1}h' = v$. But $\sigma(\overline{M})^{-1}\sigma(M) \in U$ and $\sigma(b)M^{-1}h' \in V_2$, so we deduce $\sigma(b)M^{-1}h' = v$, and finally $M^{-1}h' = h$ as claimed.

This shows that Ψ induces a well-defined map $P(k) \backslash \mathcal{Y} \rightarrow E \backslash GL_n(k)$. We leave it to the reader to check that it is bijective. \square

4.2. The canonical filtration. Let (D, F, V) be a Dieudonné space. The operators V and F^{-1} act naturally on the set of subspaces of D . It can be shown that there exists a flag of D which is stable by V and F^{-1} and which is coarsest among all such flags. This flag is called the canonical filtration of D . It is obtained by applying all finite combinations of V, F^{-1} to the flag $0 \subset D$.

Choose a basis of D and write $F = a \otimes \sigma$ and $V = b \otimes \sigma^{-1}$ with $(a, b) \in \mathcal{X}$. By choosing an appropriate basis, we will assume that $(a, b) \in \mathcal{Y}$.

Remark 4.2.1. Actually, there exists a basis such that $(a, b) \in \mathcal{Y}$ and such that the coefficients of a, b are either 0 or 1 and each column and each row has at most one non-zero coefficient.

Let $H \subset k^n$ be a subspace such that $\text{Im}(a) \oplus H = k^n$ and let $f_H \in GL_n(k)$ be the element defined in diagram (4.1.6). We have the following easy lemma:

Lemma 4.2.2. *For any subspace $W \subset k^n$, one has the following relations:*

$$(4.2.1) \quad V(W) = V_2 \cap (\sigma^{-1}(f_H^{-1}W) + V_1)$$

$$(4.2.2) \quad F^{-1}(W) = V_2 + (\sigma^{-1}(f_H^{-1}W) \cap V_1).$$

In particular, the right-hand terms are independent of the choice of H . This observation suggests the following definition:

Definition 4.2.3. For $f \in GL_n(k)$, there exists a unique coarsest flag $\mathcal{F}(f)$ of k^n satisfying the following properties:

(1) For any $W \in \mathcal{F}(f)$, the following inclusions hold

$$(4.2.3) \quad V_2 \cap (\sigma^{-1}(f^{-1}W) + V_1) \subset W \subset V_2 + (\sigma^{-1}(f^{-1}W) \cap V_1).$$

(2) For any $W \in \mathcal{F}(f)$, all subspaces appearing in (4.2.3) are in $\mathcal{F}(f)$.

The flag $\mathcal{F}(f)$ is simply the canonical flag attached to the Dieudonné space corresponding to the left-coset fV under the bijection Ψ .

Lemma 4.2.4. Let $f \in GL_n(k)$. The following assertions hold

(1) For all $v \in V$, one has $\mathcal{F}(fv) = \mathcal{F}(f)$.

(2) For $M \in P$, one has $\mathcal{F}(Mf\sigma(\overline{M})^{-1}) = M\mathcal{F}(f)$.

We leave the verification of the lemma to the reader. In particular, the conjugation class of $\mathcal{F}(f)$ depends only on the E -orbit of f . Denote by $P(f) := \text{Stab}(\mathcal{F}(f))$. Since \mathcal{F} contains $\text{Im}(V) = V_2$, we have $P(f) \subset P$. Furthermore, for $v \in V$ and $M \in P$, one has

$$(4.2.4) \quad P(fv) = P(f)$$

$$(4.2.5) \quad P(Mf\sigma(\overline{M})^{-1}) = {}^M P(f).$$

4.3. The canonical parabolic versus P_w . Denote by T the diagonal torus of $G := GL_n$, and let B be the Borel subgroup of upper-triangular matrices. The Weyl group $W(G, T)$ is the symmetric group S_n , which we identify with a subgroup of $G(k)$ by letting it act on k^n by $\tau(e_i) = e_{\tau(i)}$ for all $\tau \in S_n$ and $i \in \{1, \dots, n\}$.

Using the notations of section 1.2, define a permutation

$$(4.3.1) \quad z := w_0 w_{0,I} = \begin{pmatrix} 0 & I_{n-d} \\ I_d & 0 \end{pmatrix}.$$

Then (B, T, z) is a frame for the zip datum (G, P, Q, L, M, φ) . For $w \in {}^I W$, set $f_w := zw$. By the parametrization (1.3.1), the set $\{zw, w \in {}^I W\}$ is a set of representatives of the E -orbits in G . For $w \in {}^I W$, it is easy to see that any $W \in \mathcal{F}(f_w)$ is spanned by $(e_i)_{i \in C_W}$ for some subset $C_W \subset \{1, \dots, n\}$. In particular we have $T \subset P(f_w)$. Note that for all $w \in {}^I W$, we have simplified formulas:

$$(4.3.2) \quad V_2 \cap (\sigma^{-1}(f_w^{-1}W) + V_1) = V_2 \cap f_w^{-1}W$$

$$(4.3.3) \quad V_2 + (\sigma^{-1}(f_w^{-1}W) \cap V_1) = V_2 + f_w^{-1}W.$$

There is a unique Levi subgroup $L(f_w) \subset P(f_w)$ containing T . Finally, for $w \in {}^I W$, denote by $L_w \subset L$ and $P_w \subset P$ the subgroups defined in (1.4.1) and (1.4.3).

Proposition 4.3.1. We have $P(f_w) = P_w$ and $L(f_w) = L_w$.

Proof. We will show first that ${}^z B \subset P(f_w)$. Clearly it suffices to show ${}^z B \cap L \subset P(f_w)$. Note that since $w \in {}^I W$, we have ${}^z B \cap L = B \cap L = {}^{zw} B \cap L$. From this it follows that if $W \subset k^n$ is a subspace such that ${}^z B \cap L \subset \text{Stab}(W)$, then ${}^z B \cap L$ stabilizes also $\sigma^{-1}(f_w^{-1}(W))$. From this it follows easily by induction that ${}^z B \cap L$ stabilizes $\mathcal{F}(f_w)$, hence ${}^z B \cap L \subset P(f_w)$ as claimed.

To finish the proof, it suffices to show the second assertion. By definition we have $f_w \varphi(L_w) = L_w$. Hence if $W \subset k^n$ is a subspace such that $L_w \subset \text{Stab}(W)$ then $L_w \subset \text{Stab}(\sigma^{-1}(f_w^{-1}(W)))$. From this, it follows again by an easy induction that

L_w stabilizes $\mathcal{F}(f_w)$, so $L_w \subset P(f_w)$. Since L_w contains the torus T , we deduce that $L_w \subset L(f_w)$.

Finally, we must show that ${}^{f_w}\varphi(L(f_w)) = L(f_w)$. Since $L(f_w)$ is clearly defined over \mathbf{F}_p , this is the same as ${}^{f_w}L(f_w) = L(f_w)$. Let $k^n = D_1 \oplus \dots \oplus D_m$ denote the decomposition attached to $L(f_w)$, numbered so that the filtration $\mathcal{F}(f_w)$ is composed of the subspaces $W_j := \bigoplus_{i=1}^j D_i$ for $1 \leq j \leq m$. There exists an integer $1 \leq r \leq m$ such that $W_r = V_2$ (and then necessarily $V_1 = \bigoplus_{j=r+1}^m D_j$). We need to show that f_w permutes the $(D_i)_{1 \leq i \leq m}$. For this, it suffices to show that if $f_w(D_i) \cap D_j \neq 0$, then $D_j \subset f_w(D_i)$ for all $1 \leq i, j \leq m$.

First assume $1 \leq j \leq r$ and let $1 \leq i \leq m$ be the smallest integer such that $D_j \cap f_w(D_i) \neq 0$. We have $0 \neq D_j \cap f_w(D_i) \subset V_2 \cap f_w(W_i)$, which implies $D_j \subset V_2 \cap f_w(W_i)$. By minimality of i , we deduce $D_j \subset f_w(D_i)$.

Now assume $r < j \leq m$ and let $1 \leq i \leq m$ be the smallest integer such that $D_j \cap f_w(D_i) \neq 0$. Then $0 \neq D_j \cap f_w(D_i) \subset V_2 + f_w(W_i)$, which implies $D_j \subset (V_2 + f_w(W_i)) \cap V_1 = f_w(W_i) \cap V_1 \subset f_w(W_i)$. By minimality of i , we deduce $D_j \subset f_w(D_i)$, which terminates the proof. \square

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